

# (CO)HOMOLOGY OF ABELIAN GROUPS WITH (CO)INVARIANT COEFFICIENTS MODULES

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**ABSTRACT.** Let  $A$  be an abelian group,  $B$  a subgroup of  $A$  and let  $M$  be an  $A$ -module. We show that if  $A/B$  is finite,  $l$ -torsion and  $M$  is a  $\mathbb{Z}[1/l]$ -module, then the natural maps  $H_n(A, M_B) \rightarrow H_n(A, M_A)$ ,  $H^n(A, M^A) \rightarrow H^n(A, M^B)$  are isomorphism. We use these isomorphisms to study the homology and cohomology of special linear groups.

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## INTRODUCTION

For a group  $G$  and a  $G$ -module  $M$ , let  $M_G$  and  $M^G$  be the co-invariants and invariants of  $M$  with respect to the  $G$ -action, i.e.

$$M_G := M / \langle m - gm \mid g \in G, m \in M \rangle,$$

$$M^G := \{m \in M \mid gm = m \text{ for every } g \in G\}.$$

Clearly  $M_G$  and  $M^G$  are trivial  $G$ -modules and it is easy to see that

$$M_G \simeq M \otimes_G \mathbb{Z} \simeq H_0(G, M), \quad M^G \simeq \text{Hom}_G(\mathbb{Z}, M) \simeq H^0(G, M),$$

where we consider  $\mathbb{Z}$  as a trivial  $G$ -module.

Let  $A$  be an abelian group and  $B$  a subgroup of  $A$ . For an  $A$ -module  $M$ , the natural maps  $M_B \rightarrow M_A$  and  $M^A \rightarrow M^B$  of  $A$ -modules induce the maps

$$H_n(A, M_B) \rightarrow H_n(A, M_A), \quad H^n(A, M^A) \rightarrow H^n(A, M^B).$$

Our first main theorem claims that these maps are isomorphism if  $A/B$  is a finite  $l$ -torsion group and  $M$  has a  $\mathbb{Z}[1/l]$ -module structure.

As an application, we study the homology and cohomology group of the general and special linear groups. As our second main theorem we show that if  $R$  is a commutative ring and  $0 \leq q \leq n$ , then

$$H_q(\text{GL}_n(R), \mathbb{Z}[1/n]) \rightarrow H_q(\text{GL}(R), \mathbb{Z}[1/n])$$

is an isomorphism if and only if

$$H_q(\text{SL}_n(R), \mathbb{Z}[1/n])_{R^*} \rightarrow H_q(\text{SL}(R), \mathbb{Z}[1/n])$$

is an isomorphism. We use this to show that if  $R$  is a commutative ring with many units, e.g. a semi-local ring with infinite residue fields, then for any  $0 \leq q \leq n$  and any  $p \geq 0$  we have the isomorphism

$$H_p(R^*, H_q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])) \xrightarrow{\simeq} H_p(R^*, H_q(\mathrm{SL}(R), \mathbb{Z}[1/n])).$$

A similar results can be proved for cohomology of these groups provided that  $R^*/R^{*n}$  is finite. The homology of general and special linear groups has important applications in algebraic  $K$ -theory [4], [3] and other areas of Mathematics.

### 1. (CO)INVARIANTS AND (CO)HOMOLOGY OF GROUPS

For a group  $G$  and a  $G$ -module  $M$  consider the map  $\alpha_G : M^G \rightarrow M_G$ , given by  $m \mapsto \overline{m}$ .

**Lemma 1.1.** *Let  $G$  be a finite group,  $R$  a commutative ring and  $\mathrm{Mod}_{RG}$  the category of left  $RG$ -modules. Let  $F, F' : \mathrm{Mod}_{RG} \rightarrow \mathrm{Mod}_R$  be the functors  $F(M) := M_G$  and  $F'(M) := M^G$ , respectively.*

- (i) *If  $1/|G| \in R$ , then  $F$  and  $F'$  are exact.*
- (ii) *If  $G$  is abelian,  $l$ -torsion and  $1/l \in R$ , then  $F$  and  $F'$  are exact.*

*Proof.* Since  $F$  is right exact and  $F'$  is left exact, to prove the claims it is sufficient to prove that  $\alpha_G$  is an isomorphism. Let  $\overline{N} : M_G \rightarrow M^G$ ,  $\overline{m} \mapsto Nm$ , where  $N := \sum_{g \in G} g \in RG$ . Then clearly  $\overline{N} \circ \alpha$  and  $\alpha \circ \overline{N}$  coincide with multiplication by  $|G|$ . Thus  $\alpha_G$  is an isomorphism, which proves (i). The proof of (ii) is by induction on the size of  $G$ . Let  $H$  be a nontrivial cyclic subgroup of  $G$ . The map  $\alpha_G$  coincides with the following composition of maps

$$M^G \xrightarrow{\simeq} (M^H)^{G/H} \xrightarrow{\alpha_H} (M_H)^{G/H} \xrightarrow{\alpha_{G/H}} (M_H)_{G/H} \xrightarrow{\simeq} M_G.$$

Now the claim follows from (i) and the induction step.  $\square$

The following corollary is well-known.

**Corollary 1.2.** *Let  $G$  be a group,  $R$  a commutative ring,  $M$  an  $RG$ -module and  $n \geq 1$ .*

- (i) *If  $G$  is finite and  $R = \mathbb{Z}[1/|G|]$ , then  $H_n(G, M) = H^n(G, M) = 0$ .*
- (ii) *If  $G$  is finite, abelian,  $l$ -torsion and  $R = \mathbb{Z}[1/l]$ , then  $H_n(G, M) = H^n(G, M) = 0$ .*
- (iii) *If  $G$  is abelian,  $l$ -torsion and  $R = \mathbb{Z}[1/l]$ , then  $H_n(G, M) = 0$ .*

*Proof.* Let  $P_\bullet \rightarrow M$  and  $M \rightarrow I^\bullet$  be projective and injective resolutions of  $RG$ -module  $M$ , respectively. One can show that

$$H_n(G, M) \simeq H_n((P_\bullet)_G), \quad H^n(G, M) \simeq H^n((I^\bullet)^G)$$

(see [1, III.6, 1.4 and III.6, Exercise 1]). Now the parts (i) and (ii) follow from Lemma 1.1. To prove (iii), we write  $G$  as direct limit of its finite subgroups, e.g.  $G = \varinjlim G_i$ . Since  $H_n(G, M) \simeq \varinjlim H_n(G_i, M)$  (see Exercise 3 in Chapter V.5 from [1]), the claim follows from (ii).  $\square$

**Lemma 1.3.** *Let  $G$  be a finite group,  $R$  a commutative ring,  $M$  an  $RG$ -module and  $N$  an  $R$ -module with the trivial  $G$ -action. If  $1/|G| \in R$ , then for any  $n \geq 0$ ,*

$$\mathrm{Tor}_n^R(N, M)_G \simeq \mathrm{Tor}_n^R(N, M_G), \quad \mathrm{Ext}_R^n(N, M^G) \simeq \mathrm{Ext}_R^n(N, M)_G^G.$$

*The same claims are true if  $G$  is finite, abelian,  $l$ -torsion and  $1/l \in R$ .*

*Proof.* First we look at the functor  $\mathrm{Tor}$ . Since

$$\mathrm{Tor}_0^{\mathbb{Z}}(N, M)_G \simeq (N \otimes_{\mathbb{Z}} M) \otimes_G \mathbb{Z} \simeq N \otimes_{\mathbb{Z}} M_G \simeq \mathrm{Tor}_0^{\mathbb{Z}}(N, M_G),$$

the claim is true for  $n = 0$ . Let  $0 \rightarrow N_{n-1} \rightarrow F \rightarrow N \rightarrow 0$  be a short exact sequence of  $R$ -modules such that  $F$  is free. If  $n \geq 2$ , from the long exact sequence, we get the isomorphism  $\mathrm{Tor}_n^R(N, M) \simeq \mathrm{Tor}_{n-1}^R(N_{n-1}, M)$ . If we continue this process, we will find an  $R$ -module  $N_1$  such that

$$\mathrm{Tor}_n^R(N, M) \simeq \mathrm{Tor}_1^R(N_1, M).$$

So it is sufficient to proof the theorem for  $n = 1$ . From an exact sequence  $0 \rightarrow N_1 \rightarrow F \rightarrow N \rightarrow 0$ ,  $F$  free, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Tor}_1^R(N, M) & \longrightarrow & N_1 \otimes_R M & \longrightarrow & F \otimes_R M & \longrightarrow & N \otimes_R M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Tor}_1^R(N, M_G) & \longrightarrow & N_1 \otimes_R M_G & \longrightarrow & F \otimes_R M_G & \longrightarrow & N \otimes_R M_G & \longrightarrow & 0. \end{array}$$

From Lemma 1.1, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Tor}_1^R(N, M)_G & \longrightarrow & (N_1 \otimes_R M)_G & \longrightarrow & (F \otimes_R M)_G & \longrightarrow & (N \otimes_R M)_G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Tor}_1^R(N, M_G) & \longrightarrow & N_1 \otimes_R M_G & \longrightarrow & F \otimes_R M_G & \longrightarrow & N \otimes_R M_G & \longrightarrow & 0. \end{array}$$

Now the claim follows from the five lemma. The proof of the claim for the functor  $\mathrm{Ext}$  is similar. In fact here we should use an exact sequence  $0 \rightarrow N \rightarrow I \rightarrow N^1 \rightarrow 0$ , where  $I$  is an injective  $R$ -module.  $\square$

*Example 1.4.* By a simple example we show that the condition  $1/|G| \in R$  can not be removed from the above lemma. Let  $M = \mathbb{Z}$  and  $G := \{1, g\}$  be the cyclic group of order two. If we define the action of  $G$  on  $M = \mathbb{Z}$  as  $g.m := -m$ , then  $M_G = \mathbb{Z}/2$  and  $M^G = 0$ . Thus we have

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, M)_G = 0, \quad \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, M_G) = \mathbb{Z}/2,$$

and

$$\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/2, M^G) = 0, \quad \mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/2, M)_G^G = \mathbb{Z}/2.$$

Let  $H$  be a normal subgroup of  $G$  and let  $M_H \rightarrow M_G$  and  $M^G \rightarrow M^H$  be the natural maps of  $G$ -modules. The following theorem is our first main result.

**Theorem 1.5.** *Let  $A$  be an abelian group and  $B$  a subgroup of  $A$  such that  $A/B$  is finite and  $l$ -torsion. Let  $R = \mathbb{Z}[1/l]$  and  $M$  be an  $RA$ -module. Then for any  $n \geq 0$ ,*

$$H_n(A, M_B) \simeq H_n(A, M_A), \quad H^n(A, M^A) \simeq H^n(A, M^B).$$

*Proof.* By the Universal Coefficient Theorem we have the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_n(B, R) \otimes_R M_B & \longrightarrow & H_n(B, M_B) & \longrightarrow & \text{Tor}_1^R(H_{n-1}(B, R), M_B) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_n(B, R) \otimes_R M_A & \longrightarrow & H_n(B, M_A) & \longrightarrow & \text{Tor}_1^R(H_{n-1}(B, R), M_A) \longrightarrow 0. \end{array}$$

From this and Lemma 1.1 we obtain the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & (H_n(B, R) \otimes_R M_B)_{A/B} & \longrightarrow & H_n(B, M_B)_{A/B} & \longrightarrow & \text{Tor}_1^R(H_{n-1}(B, R), M_B)_{A/B} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H_n(B, R) \otimes_R M_A & \longrightarrow & H_n(B, M_A) & \longrightarrow & \text{Tor}_1^R(H_{n-1}(B, R), M_A) \longrightarrow 0. \end{array}$$

Since the action of  $A/B$  on  $H_n(B, R)$  is trivial, by Lemma 1.3 the left and right hand side column maps of this diagram are isomorphism. Therefore  $H_n(B, M_B)_{A/B} \simeq H_n(B, M_A)$  and thus

$$(1.1) \quad H_n(B, M_B)_A \simeq H_n(B, M_B)_{A/B} \simeq H_n(B, M_A).$$

From the exact sequence  $1 \rightarrow B \rightarrow A \rightarrow A/B \rightarrow 1$  and the natural map  $M_B \rightarrow M_A$ , we obtain the morphism of Lyndon/Hochschild-Serre spectral sequences

$$\begin{array}{ccc} E_{p,q}^2 = H_p(A/B, H_q(B, M_B)) & \Rightarrow & H_{p+q}(A, M_B) \\ \downarrow & & \downarrow \\ E_{p,q}'^2 = H_p(A/B, H_q(B, M_A)) & \Rightarrow & H_{p+q}(A, M_A). \end{array}$$

By Corollary 1.2, for any  $p \geq 1$ ,  $E_{p,q}^2 = 0 = E_{p,q}'^2$ . Now by an easy analysis of the above spectral sequences for any  $n \geq 0$ , we obtain the commutative diagram

$$\begin{array}{ccc} E_{0,n}^2 = H_0(A/B, H_n(B, M_B)) & \xrightarrow{\simeq} & H_n(A, M_B) \\ \downarrow & & \downarrow \\ E_{0,n}'^2 = H_0(A/B, H_n(B, M_A)) & \xrightarrow{\simeq} & H_n(A, M_A). \end{array}$$

Since the action of  $A$  on  $H_n(A, M_A)$  and the action of  $B$  on  $H_n(B, M_B)$  is trivial,  $E_{0,n}^2 \simeq H_n(B, M_B)_A$  and  $E_{0,n}'^2 \simeq H_n(B, M_A)$ . Now by the isomorphism (1.1), we have  $E_{0,n}^2 \simeq E_{0,n}'^2$ . Therefore the isomorphism  $H_n(A, M_B) \simeq H_n(A, M_A)$  follows from the above commutative diagram. The other isomorphism can be proved in a similar way. In fact, first one should prove

that  $H^n(B, M^A) \simeq H^n(B, M^B)^A$  and use it to prove the isomorphism  $H^n(A, M^A) \simeq H^n(A, M^B)$ , as in above.  $\square$

**Corollary 1.6.** *Let  $A$  be an abelian group and  $B$  a subgroup of  $A$  such that  $A/B$  is  $l$ -torsion. Let  $R = \mathbb{Z}[1/l]$  and  $M$  be an  $RA$ -module. Then for any  $n \geq 0$ ,  $H_n(A, M_B) \simeq H_n(A, M_A)$ . In particular if the action of  $B$  on  $M$  is trivial, then*

$$H_n(A, M) \simeq H_n(A, M_A).$$

*Proof.* The group  $A/B$  can be written as direct limit of its finite subgroups, e.g.  $A/B = \varinjlim A_i/B$ . Since,

$$H_n(A, M_B) \simeq \varinjlim H_n(A_i, M_B), \quad H_n(A, M_A) \simeq \varinjlim H_n(A_i, M_{A_i})$$

(see Exercise 3 in Chapter V.5 from [1]), we may assume that  $A/B$  is finite, which the claim follows from Theorem 1.5.  $\square$

## 2. (CO)HOMOLOGY OF SPECIAL LINEAR GROUPS

In this section we give an application of Theorem 1.5 and Corollary 1.6. Let  $R$  be a commutative ring. The conjugate action of  $R^*$  on  $\mathrm{SL}_n(R)$ , given by

$$a.A := \mathrm{diag}(a, I_{n-1}).A.\mathrm{diag}(a^{-1}, I_{n-1}),$$

induces a natural action of  $R^*$  on  $H_q(\mathrm{SL}_n(R), \mathbb{Z})$ . Since

$$\begin{aligned} a^n.A &= \mathrm{diag}(a^n, I_{n-1}).A.\mathrm{diag}(a^{-n}, I_{n-1}) \\ &= \mathrm{diag}(a^{n-1}, a^{-1}I_{n-1}).aI_n.A.a^{-1}I_n.\mathrm{diag}(a^{-(n-1)}, aI_{n-1}) \\ &= \mathrm{diag}(a^{n-1}, a^{-1}I_{n-1}).A.\mathrm{diag}(a^{-(n-1)}, aI_{n-1}), \end{aligned}$$

the action of  $R^{*n}$  on  $H_q(\mathrm{SL}_n(R), \mathbb{Z})$  is trivial [1, Chap. II, Proposition 6.2]. Since  $R^*/R^{*n}$  is an  $n$ -torsion group, by Corollary 1.6

$$(2.1) \quad H_p(R^*, H_q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])) \xrightarrow{\simeq} H_p(R^*, H_q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])_{R^*}).$$

If  $R^*/R^{*n}$  is finite, then by Theorem 1.5

$$H^p(R^*, H^q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])^{R^*}) \xrightarrow{\simeq} H^p(R^*, H^q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])).$$

Consider the natural inclusion  $\mathrm{SL}_n(R) \rightarrow \mathrm{SL}(R)$ . It is easy to see that the action of  $R^*$  on  $H_q(\mathrm{SL}(R), \mathbb{Z})$  and  $H^q(\mathrm{SL}(R), \mathbb{Z})$  is trivial. Thus we have the natural maps

$$H_q(\mathrm{SL}_n(R), \mathbb{Z})_{R^*} \rightarrow H_q(\mathrm{SL}(R), \mathbb{Z}), \quad H^q(\mathrm{SL}(R), \mathbb{Z}) \rightarrow H^q(\mathrm{SL}_n(R), \mathbb{Z})^{R^*}.$$

**Proposition 2.1.** *Let  $R$  be a commutative ring and  $q \leq n$ .*

(i) *The map  $H_q(\mathrm{GL}_n(R), \mathbb{Z}[1/n]) \rightarrow H_q(\mathrm{GL}(R), \mathbb{Z}[1/n])$  is an isomorphism if and only if  $H_q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])_{R^*} \rightarrow H_q(\mathrm{SL}(R), \mathbb{Z}[1/n])$  is an isomorphism.*

(ii) *If  $R^*/R^{*n}$  is finite, then  $H^q(\mathrm{GL}(R), \mathbb{Z}[1/n]) \rightarrow H^q(\mathrm{GL}_n(R), \mathbb{Z}[1/n])$  is an isomorphism if and only if  $H^q(\mathrm{SL}(R), \mathbb{Z}[1/n]) \rightarrow H^q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])^{R^*}$  is an isomorphism.*

*Proof.* From the map of extensions

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{SL}_n(R) & \longrightarrow & \mathrm{GL}_n(R) & \longrightarrow & R^* \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathrm{SL}(R) & \longrightarrow & \mathrm{GL}(R) & \longrightarrow & R^* \longrightarrow 1, \end{array}$$

we get the morphism of Lyndon/Hochschild-Serre spectral sequences

$$\begin{array}{ccc} E_{p,q}^2 = H_p(R^*, H_q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])) & \Rightarrow & H_{p+q}(\mathrm{GL}_n(R), \mathbb{Z}[1/n]) \\ \downarrow & & \downarrow \\ E_{p,q}'^2 = H_p(R^*, H_q(\mathrm{SL}(R), \mathbb{Z}[1/n])) & \Rightarrow & H_{p+q}(\mathrm{GL}(R), \mathbb{Z}[1/n]). \end{array}$$

This gives us a map of filtration

$$\begin{array}{ccccccccccc} 0 = F_{-1} & \subseteq & F_0 & \subseteq & \cdots & \subseteq & F_{n-1} & \subseteq & F_n = H_n(\mathrm{GL}_n(R), \mathbb{Z}[1/n]) \\ & & \downarrow & & & & \downarrow & & \downarrow \\ 0 = F'_{-1} & \subseteq & F'_0 & \subseteq & \cdots & \subseteq & F'_{n-1} & \subseteq & F'_n = H_n(\mathrm{GL}(R), \mathbb{Z}[1/n]), \end{array}$$

such that  $E_{i,n-i}^\infty \simeq F_i/F_{i-1}$ ,  $E_{i,n-i}'^\infty \simeq F'_i/F'_{i-1}$ .

If  $H_q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])_{R^*} \simeq H_q(\mathrm{SL}(R), \mathbb{Z}[1/n])$ ,  $q \leq n$ , then by (2.1), we have  $E_{p,q}^2 \simeq E_{p,q}'^2$ , where  $p \geq 0$  and  $q \leq n$ . Thus the isomorphism  $H_q(\mathrm{GL}_n(R), \mathbb{Z}[1/n]) \simeq H_q(\mathrm{GL}(R), \mathbb{Z}[1/n])$  follows from the above morphism of filtration.

Conversely, let  $H_q(\mathrm{GL}_n(R), \mathbb{Z}[1/n]) \simeq H_q(\mathrm{GL}(R), \mathbb{Z}[1/n])$ ,  $q \leq n$ . By induction on  $q$ , we prove that for any  $p \geq 0$ ,

$$(2.2) \quad H_p(R^*, H_q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])) \simeq H_p(R^*, H_q(\mathrm{SL}(R), \mathbb{Z}[1/n])).$$

If  $q = 0$ , then the claim is trivial. So let  $q \geq 1$  and assume that the claim is true for  $q \leq n-1$ . By induction hypothesis for  $q \leq n-1$ ,  $E_{p,q}^2 \simeq E_{p,q}'^2$ . From an easy comparison of the above spectral sequences it follows that for  $p+q \leq n$  and  $q \leq n-1$ ,

$$E_{p,q}^\infty \simeq E_{p,q}'^\infty.$$

These imply that for all  $i$ ,  $F_i \simeq F'_i$ . The map  $\delta : \mathrm{GL}(R) \rightarrow \mathrm{SL}(R)$  given by  $B \mapsto \begin{pmatrix} \det(B)^{-1} & 0 \\ 0 & B \end{pmatrix}$  induces a homomorphism  $\delta_* : H_i(\mathrm{GL}(R), \mathbb{Z}) \rightarrow H_i(\mathrm{SL}(R), \mathbb{Z})$  such that the composition

$$H_i(\mathrm{SL}(R), \mathbb{Z}) \xrightarrow{\mathrm{inc}_*} H_i(\mathrm{GL}(R), \mathbb{Z}) \xrightarrow{\delta_*} H_i(\mathrm{SL}(R), \mathbb{Z})$$

is the identity map. Therefore for any  $i \geq 0$ ,  $H_i(\mathrm{SL}(R), \mathbb{Z})$  embeds in  $H_i(\mathrm{GL}(R), \mathbb{Z})$ . Now the injectivity of the homomorphism

$$H_n(\mathrm{SL}(R), \mathbb{Z}[1/n]) \rightarrow H_n(\mathrm{GL}(R), \mathbb{Z}[1/n])$$

implies that  $F'_0 = H_0(R^*, H_n(\mathrm{SL}(R), \mathbb{Z}[1/n])) = H_n(\mathrm{SL}(R), \mathbb{Z}[1/n])$ . On the other hand  $F_0 = H_0(R^*, H_n(\mathrm{SL}_n(R), \mathbb{Z}[1/n]))/T$  for some subgroup  $T$ . From the map

$$\gamma : R^* \times \mathrm{SL}_n(R) \rightarrow \mathrm{GL}_n(R), \quad (b, B) \mapsto bB,$$

we get the exact sequence

$$1 \rightarrow \mu_n(R) \rightarrow R^* \times \mathrm{SL}_n(R) \rightarrow \mathrm{GL}_n(R) \rightarrow R^*/R^{*n} \rightarrow 1,$$

which gives us two extension of groups

$$\begin{aligned} 1 \rightarrow \mu_n(R) \rightarrow R^* \times \mathrm{SL}_n(R) &\rightarrow \mathrm{im}(\gamma) \rightarrow 1, \\ 1 \rightarrow \mathrm{im}(\gamma) \rightarrow R^* \times \mathrm{GL}_n(R) &\rightarrow R^*/R^{*n} \rightarrow 1. \end{aligned}$$

Writing the Lyndon/Hochschild-Serre spectral sequence of the above exact sequences and carrying out not difficult analysis, using Corollary ??, one gets the isomorphisms

$$(2.3) \quad H_n(\mathrm{im}(\gamma), \mathbb{Z}[1/n]) \simeq H_n(R^* \times \mathrm{SL}_n(R), \mathbb{Z}[1/n]),$$

$$(2.4) \quad H_0(R^*/R^{*n}, H_n(\mathrm{im}(\gamma), \mathbb{Z}[1/n])) \simeq H_n(\mathrm{GL}_n(R), \mathbb{Z}[1/n]).$$

The action of  $R^{*n}$  on  $H_n(\mathrm{im}(\gamma), \mathbb{Z}[1/n])$  is trivial, so from (2.4) we obtain

$$(2.5) \quad H_0(R^*, H_n(\mathrm{im}(\gamma), \mathbb{Z}[1/n])) \simeq H_n(\mathrm{GL}_n(R), \mathbb{Z}[1/n]).$$

Relations (2.3) and (2.5) imply

$$H_n(R^* \times \mathrm{SL}_n(R), \mathbb{Z}[1/n])_{R^*} \simeq H_n(\mathrm{GL}_n(R), \mathbb{Z}[1/n]).$$

Now applying the Künneth formula we see that  $H_0(R^*, H_n(\mathrm{SL}_n(R), \mathbb{Z}[1/n]))$  embeds in  $H_n(\mathrm{GL}_n(R), \mathbb{Z}[1/n])$  and thus  $T = 0$ . Therefore

$$H_0(R^*, H_n(\mathrm{SL}_n(R), \mathbb{Z}[1/n])) = F_0 \simeq F'_0 = H_n(\mathrm{SL}_n(R), \mathbb{Z}[1/n]).$$

Now by (2.1) we obtain the isomorphism (2.2). This completes the induction steps and thus the proof of the isomorphism

$$H_n(\mathrm{SL}_n(R), \mathbb{Z}[1/n])_{R^*} \simeq H_n(\mathrm{SL}_n(R), \mathbb{Z}[1/n]).$$

Thus we have proved the part (i). The part (ii) can be proved in a similar way.  $\square$

We say that a commutative ring  $R$  is a *ring with many units* if for any  $n \geq 2$  and for any finite number of surjective linear forms  $f_i : R^n \rightarrow R$ , there exists a  $v \in R^n$  such that, for all  $i$ ,  $f_i(v) \in R^*$ . Important examples of rings with many units are semi-local rings with infinite residue fields. For more about these rings please see [3, Section 2], [2, Section 1]. The following theorem is due to Suslin.

**Theorem 2.2** (Homological stability for general linear groups). *Let  $R$  be a commutative ring with many units. Then the map*

$$H_l(\mathrm{GL}_n(R), \mathbb{Z}) \rightarrow H_l(\mathrm{GL}_{n+1}(R), \mathbb{Z})$$

*is an isomorphism if  $l \leq n$ .*

*Proof.* See [5, Theorem 3.4] and [2, Theorem 1].  $\square$

**Corollary 2.3.** *Let  $q \leq n$  be nonnegative integers and let  $R$  be a ring with many units. Then for any  $p \geq 0$ , the natural inclusion  $\mathrm{SL}_n(R) \rightarrow \mathrm{SL}(R)$  induces the isomorphism*

$$H_p(R^*, H_q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])) \xrightarrow{\simeq} H_p(R^*, H_q(\mathrm{SL}(R), \mathbb{Z}[1/n])).$$

*If  $R^*/R^{*n}$  is finite, we have the isomorphism*

$$H^p(R^*, H^q(\mathrm{SL}(R), \mathbb{Z}[1/n])) \xrightarrow{\simeq} H^p(R^*, H^q(\mathrm{SL}_n(R), \mathbb{Z}[1/n])).$$

*Proof.* The claim follows from Theorem 2.2, Proposition 2.1 and 2.1.  $\square$

*Remark 2.4.* Corollary 2.3 has appeared and applied in [3, Lemma 2]. But the proof given there has a gap. This article grew out of our attempt to fill this gap and also to generalize it.

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